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Positive Solutions and Asymptotic Behavior of Delay Differential Equations with Nonlinear Impulses*

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Consider the delay differential equation (DDE) with nonlinear impulses

$$\dot{x}(t) + \sum_{i=1}^{n} p_i(t)x(t-\tau_i) = 0, \qquad t \neq t_j,\ t \geq t_0,$$

$$x\left(t_j^+\right) - x\left(t_j\right) = I_j\big(x\left(t_j\right)\big), \qquad j = 1, 2, \dots, \tag{$*$}$$

where $p_i \in C([t_0, \infty), \mathbf{R}^+)$, $\tau_i \geq 0$ for $i = 1, 2, \dots, n$, and $I_j \in C(\mathbf{R}, \mathbf{R})$ for $j = 1, 2, \dots$. The purpose of this paper is to obtain a necessary and sufficient condition for the existence of positive solutions of DDE without impulses and to establish a kind of order persistence of solutions of Eq. $(*)$ and criteria of the asymptotic behavior of Eq. $(*)$, which can be used to improve and develop some of the known results in the literature. © 1997 Academic Press

1. INTRODUCTION

There have been many papers considering the delay differential equation

$$\dot{x}(t) + \sum_{i=1}^{n} p_i(t)x(t-\tau_i) = 0, \tag{1}$$

where $p_i \in C([t_0, \infty), \mathbf{R}^+)$, $\tau_i \geq 0$ for $i = 1, 2, \dots, n$; see, for example, [1, 4, 5] and the references cited in [4]. There are only a few papers concerned with the impulsive delay differential equations, which is an important mathematical model of many evolution process; see [2, 3, 6, 7].

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In this paper, we consider the delay differential equations (DDE) with nonlinear impulses,

$$\dot{x}(t) + \sum_{i=1}^{n} p_i(t) x(t - \tau_i) = 0, \qquad t \neq t_j, t \geq t_0,$$
$$x(t_j^+) - x(t_j) = I_j\big(x(t_j)\big), \qquad j = 1, 2, \dots, \tag{2}$$

where $p_i \in C([t_0, \infty), \mathbf{R}^+)$, $\tau_i \geq 0$ for $i = 1, 2, \dots, n$, and $I_j \in C(\mathbf{R}, \mathbf{R})$ for $j = 1, 2, \dots$. Let $\tau = \max\{\tau_1, \tau_2, \dots, \tau_n\}$ and $t_0 < t_1 < t_2 < \cdots$, $t_j \to \infty$ as $j \to \infty$. Set

$$PC = \{\phi \colon [-\tau, 0] \to \mathbf{R} \text{ is piecewise left continuous}\}.$$

With Eq. (2), one associates an initial condition of the form

$$x_\sigma = \phi(s), \qquad s \in [-\tau, 0], \tag{3}$$

where $x_\sigma = x(\sigma + s)$ and $\phi \in PC$.

Definition 1. $x(t)$ is called a solution corresponding to t_0 of Eq. (2) if $x\colon [t_0 - \tau, \infty) \to \mathbf{R}$ is continuous for $t \neq t_j$; x is continuously differentiable for $t \geq t_0$, $t \neq t_j$, and $t \neq t_j + \tau_i$, and $x(t_j^+), x(t_j^-)$ exist and $x(t_j^-) = x(t_j)$, $j = 1, 2, \dots,$ and x satisfies Eq. (2).

Definition 2. $x(t)$ is called a solution corresponding to t_0 of the initial value problem (2) and (3) if x is a solution corresponding to t_0 of Eq. (2) and satisfies (3), denoted by $x(t_0, \phi)(t)$.

Definition 3. $x(t_0, \phi)(t)$ is called a positive solution of Eq. (2) if $\phi \geq 0$, $s \in [-\tau, 0]$, and $x(t_0, \phi)(t) > 0$, $t \geq t_0$.

Definition 4. A solution $x(t)$ of Eq. (2) is said to be nonoscillatory if it is eventually positive or eventually negative; otherwise it will be called oscillatory.

Analogously, we can define the various solutions of Eq. (1).

Our aim in this paper is to obtain results on existence and asymptotic behavior of nonoscillatory solutions of Eq. (2). This paper is organized as follows. In Section 2, we show a necessary and sufficient condition ensuring that there exist positive solutions in Eq. (1), the special case of which includes the corresponding results in [4]. Furthermore, under the conditions we establish in Section 3, several new results which assert that there exists a kind of order relation between the solutions of Eq. (1) and those of Eq. (2), that is, the differences between the solutions of (1) and the corresponding ones of (2), are always positive or negative. Finally, some criteria of asymptotic behavior of Eq. (2) are given in Section 4, which improve and generalize the known criteria [7].

2. POSITIVE SOLUTIONS OF EQ. (1)

Let $h_i(t) = \min\{t_0, t - \tau_i\}$, $H_i(t) = \max\{t_0, t - \tau_i\}$, and

$$\Phi = \{\phi \in C([-\tau, 0], \mathbf{R}^+) \mid \phi(0) > 0 \text{ and } \phi(s) \le \phi(0), -\tau \le s \le 0\}.$$

Our aim in this section is to establish the following necessary and sufficient condition for the positive solutions of Eq. (1)

THEOREM 1. *For Eq.* (1), *the following statements are equivalent*:

(a) *There exist* $k_i(t) \in C([t_0, \infty), \mathbf{R}^+)$, $i = 1, 2, \dots, n$, *such that*

$$\int_{H_j(t)}^{t} \sum_{i=1}^{n} p_i(s) k_i(s)\, ds \le \ln k_j(t), \qquad t \ge t_0,\ j = 1, 2, \dots, n. \tag{4}$$

(b) *For* $\forall\, \phi \in \Phi$, $x(t_0, \phi)(t)$ *is a positive solution of Eqs.* (1) *and* (3).

Proof. (a) $\Rightarrow$ (b) Assume that (a) holds. Set

$$\beta(t) = -\sum_{i=1}^{n} p_i(t) k_i(t), \qquad \gamma(t) = 0, \qquad \text{for } t \ge t_0.$$

For all $\delta \in C([t_0, \infty), \mathbf{R})$ such that $\beta(t) \le \delta(t) \le \gamma(t)$, $t \ge t_0$, it follows from (4) that

$$-\int_{H_j(t)}^{t} \delta(s)\, ds \le \int_{H_j(t)}^{t} \sum_{i=1}^{n} p_i(s) k_i(s)\, ds \le \ln k_j(t),$$

where $t \ge t_0$, $j = 1, 2, \dots, n$. Hence, for $\forall\, \phi \in \Phi$, we have

$$\begin{aligned}\gamma(t) = 0 &\ge -\sum_{i=1}^{n} p_i(t) \frac{\phi(h_i(t) - t_0)}{\phi(0)} \exp\left(-\int_{H_i(t)}^{t} \delta(s)\, ds\right)\\ &\ge -\sum_{i=1}^{n} p_i(t) k_i(t) = \beta(t).\end{aligned}$$

In view of Theorem 3.1.1 in [4], $x(t_0, \phi)(t)$ is a positive solution corresponding to t_0 of Eqs. (1) and (3).

(b) $\Rightarrow$ (a) Assume that $y(t_0, \phi)(t)$ is a positive solution of Eq. (1). Let $y(t) = y(t_0, \phi)(t)$ and

$$\alpha(t) = \frac{\dot{y}(t)}{y(t)}, \qquad t \ge t_0.$$

So

$$y(t) = \phi(0) \exp\left(\int_{t_0}^{t} \alpha(s)\, ds\right), \qquad t \ge t_0.$$

Hence, it follows from Theorem 3.1.1 in [4] that

$$\alpha(t) + \sum_{i=1}^{n} p_i(t) \frac{\phi(h_i(t) - t_0)}{\phi(0)} \exp\left(\int_{H_i(t)}^{t} \alpha(s)\, ds \right) = 0. \tag{5}$$

Set

$$k_i(t) = \exp\left(- \int_{H_i(t)}^{t} \alpha(s) ds \right).$$

Integrating (5) from $H_i(t)$ to t, we get

$$\begin{aligned} &\int_{H_j(t)}^{t} \sum_{i=1}^{n} p_i(s) \frac{\phi(h_i(s) - t_0)}{\phi(0)} k_i(s)\, ds \\ &= \int_{H_j(t)}^{t} \sum_{i=1}^{n} p_i(s) k_i(s)\, ds = - \int_{H_j(t)}^{t} \alpha(s)\, ds \\ &= \ln k_j(t), \qquad t \ge t_0,\ j = 1, 2, \dots, n \end{aligned}$$

which completes the proof of Theorem 1.

Remark 1. If $k_i(t) \equiv e$, it is easy from (4) to see that

$$\int_{H_j(t)}^{t} \sum_{i=1}^{n} p_i(s)\, ds \le \frac{1}{e}, \qquad t \ge t_0,$$

which is a well-known result.

3. SOME PROPERTIES OF SOLUTIONS OF EQ. (2)

This section is devoted to the existence of positive and nonoscillatory solutions of Eq. (2), which are based on our comparison result established here. It is interesting in itself and needed in the following lemma, which is a clear extension of Lemma 1 in [5].

LEMMA 1. *Assume that $z(t)$ is a positive solution of Eq.* (1) *and $x(t)$ is any solution of Eq.* (2). *Set*

$$w(t) = x(t)/z(t), \qquad t \ge t_0.$$

Then for $t \ge t_0$, $t \ne t_j$, $t \ne t_j + \tau_i$, $j = 1, 2, \dots,$ and $i = 1, 2, \dots, n$,

$$\dot{w}(t) = \sum_{i=1}^{n} p_i(t) \frac{z(t - \tau_i)}{z(t)} \big(w(t) - w(t - \tau_i) \big). \tag{6}$$

THEOREM 2. *Assume that*

$$p_i(t) \geq 0 \qquad \text{for } i = 1, 2, \ldots, n \quad \text{and} \quad \sum_{i=1}^{n} p_i(t) > 0, \qquad t \geq t_0, \quad (7)$$

and Eq. (1) *has a positive solution* $z(t_0, \varphi)(t)$ *corresponding to* t_0. *Then for* $\phi \in PC$ *such that*

$$\phi = \begin{cases} 0, & -\tau \leq s < 0, \\ b, & s = 0, \end{cases} \tag{8}$$

where $b \neq 0$,

$$bx(\sigma, \phi)(t) > 0, \qquad t \geq \sigma,$$

where $x(\sigma, \phi)(t)$ *is a solution corresponding to* σ *with* $\sigma \geq t_0$ *of Eqs.* (1) *and* (3).

Proof. By Theorem 1, without loss of generality, assume that $z(t) > 0$ for $t \geq t_0 - \tau$. Consider the case of $b > 0$ (the case of $b < 0$ can be treated in a similar way). Since Eq. (1) is a linear equation, it is easy to see that there exists a constant $C > 0$ such that $Cz(t)$ is a positive solution of Eq. (1) and satisfies

$$Cz(\sigma) = b. \tag{9}$$

We have to prove $x(\sigma, \phi)(t) > 0$ for $t \geq \sigma$. Note that $Cz(t) \geq \phi$ for $t \in [\sigma - \tau, \sigma]$ and it follows from Eq. (1) that there exists a constant $T > \sigma$ such that

$$x(\sigma, \phi)(t) \geq Cz(t), \qquad \sigma \leq t \leq T. \tag{10}$$

We claim that $T = \infty$. Otherwise, there exists $\bar{t} > \sigma$ such that

$$x(\sigma, \phi)(\bar{t}) = Cz(\bar{t}).$$

In view of (9) and (10), we can choose

$$t^* = \inf_{\sigma \leq t \leq \bar{t}} \{\dot{w}(t) = 0 \text{ and } \dot{w}(s) \neq 0, \ \sigma \leq s \leq t\}$$

such that

$$\dot{w}(t^*) = 0 \quad \text{and} \quad w(t^*) > w(s), \qquad \sigma - \tau \leq s < t^*,$$

where $w(t) = x(t)/Cz(t)$. Then substituting $t = t^*$ into (6), Lemma 1 and (7) imply a contradiction which completes our proof.

Remark 2. If $z(t_0, \phi)(t)$ in the proof of Theorem 2 is a positive solution of Eqs. (1) and (3) with $\phi \in PC$, the theorem is also true.

In the following text, we will suppose that if $x(t_0, \phi)(t)$ is a solution of Eqs. (1) and (3), $x^*(t_0, \phi)(t)$ is a solution of Eqs. (2) and (3).

THEOREM 3. *Suppose that the hypotheses of Theorem* 2 *hold and*

$$xI_j(x) \geq 0, \qquad j = 1, 2, \dots . \tag{11}$$

Then $z^*(t_0, \varphi)(t) \geq z(t_0, \varphi)(t)$, $t \geq t_0$.

Proof. Let $z_k(t_0, \varphi)(t)$, $k = 1, 2, \dots,$ be solutions of the equations

$$\dot{x}(t) + \sum_{i=1}^{n} p_i(t) x(t - \tau_i) = 0, \qquad t \neq t_j, t \geq t_0$$

$$x(t_j^+) - x(t_j) = I_j\big(x(t_j)\big), \qquad j = 1, 2, \dots, k,$$

and $\varphi \in PC$. Since Eq. (1) and the first equation above are linear equations, $z_1(t_0, \varphi)(t) - z(t_0, \varphi)(t)$ is a solution corresponding to t_1 of the initial problem Eqs. (1) and (3), denoted by $z_1(t_1, \phi)(t)$, where

$$\phi = \begin{cases} 0, & -\tau \leq s < 0, \\ I_1(x(t_1)), & s = 0. \end{cases}$$

Hence, Theorem 2 and (11) imply

$$z_1(t_1, \phi)(t) > 0, \qquad t > t_1.$$

So

$$z_1(t_0, \varphi)(t) \geq z(t_0, \varphi)(t), \qquad t \geq t_0.$$

By induction, for $t \geq t_0$,

$$0 < z(t_0, \varphi)(t) \leq z_1(t_0, \varphi)(t) \leq \cdots \leq z^*(t_0, \varphi)(t).$$

The proof of Theorem 3 is completed.

Applying Theorem 2 and the method used in Theorem 3, we can similarly obtain the following results.

THEOREM 4. *Assume that* (4) *and* (7) *are satisfied. Let* $x(t_0, \varphi)(t)$ *be any solution of Eqs*. (1) *and* (3). *Then for* $t \geq t_0$,

(1) *if* $I_j(x) \geq 0$, $j = 1, 2, \dots,$ $x^*(t_0, \varphi)(t) \geq x(t_0, \varphi)(t)$;

(2) *if* $I_j(x) \leq 0$, $j = 1, 2, \dots,$ $x^*(t_0, \varphi)(t) \leq x(t_0, \varphi)(t)$.

COROLLARY 1. *In Theorem* 4, *if* $I_j(x) \geq 0$ [*or* $I_j(x) \leq 0$] *and* $x(t_0, \varphi)(t)$ *is eventually positive* (*or negative*), *then* $x^*(t_0, \varphi)(t)$ *is also eventually positive* (*or negative*).

4. ASYMPTOTIC BEHAVIOR OF EQ. (2)

THEOREM 5. *Assume that Eq.* (2) *satisfies the conditions*

$$\int_{t_0}^{\infty} \sum_{i=1}^{n} p_i(s)\, ds = \infty, \tag{12}$$

$$\left| I_j(x) \right| \leq b_j |x| \quad \text{and} \quad \sum_{j=1}^{\infty} b_j < \infty, \qquad j = 1, 2, \dots, \tag{13}$$

where $b_j \geq 0$. *Then every nonoscillatory solution of Eq.* (2) *tends to zero as* $t \to \infty$.

Proof. Without loss of generality, assume that $z(t)$ is an eventually positive solution of Eq. (2). Take a sequence $\{t_k^*\}$ of $\{t_j\}_1^\infty$ such that $I_k(z(t_k^*)) > 0$ and choose a corresponding sequence $\{b_k^*\}$ from $\{b_j\}_1^\infty$. Therefore, there exists a sufficiently large $T \geq t_0$, such that

$$z(t) > 0 \quad \text{and} \quad \dot{z}(t) \leq 0, \qquad t \neq t_j,\, t \geq T, \tag{14}$$

that is, $z(t)$ is decreasing in $(t_j, t_{j+1}]$ for $t_j \geq T$, $j = m, m+1, \dots,$ with $m \in \mathbf{N}$. It is easy to see that $z(t)$ is also decreasing in $(t_k^*, t_{k+1}^*]$ for $t_k^* \geq T$, $k \geq m$. Hence for $t \geq t_k^*$,

$$\begin{aligned} z(t) &\leq z\left(t_k^{*+}\right) \leq (1 + b_k^*) z(t_k^*) \leq (1 + b_k^*) z\left(t_{k-1}^{*+}\right) \\ &\leq (1 + b_k^*)(1 + b_{k-1}^*) z(t_{k-1}^*) \\ &\leq (1 + b_k^*)(1 + b_{k-1}^*) \cdots (1 + b_m^*) z(t_m^*). \end{aligned} \tag{15}$$

In view of (13), $0 < \prod(1 + b_k^*) < \infty$. Thus, from (15), there exists a constant $M > 0$ such that

$$z(t) < M \qquad \text{for } t \geq T.$$

Now, we claim that $\liminf_{t \to \infty} z(t) = 0$. Otherwise, set

$$\liminf_{t \to \infty} z(t) = l > 0. \tag{16}$$

Then there exists $T_1 \geq T$, such that $z(t) \geq l/2$ for $t - \tau > T_1$. So

$$0 = \dot{z}(t) + \sum_{i=1}^{n} p_i(t) z(t - \tau_i)$$
$$\geq \dot{z}(t) + \frac{l}{2} \sum_{i=1}^{n} p_i(t).$$

Integration from t to ∞ with $t \geq T_1$ yields

$$l - M \sum_{k \geq m} b_k^* - z(t) + \frac{l}{2} \int_t^{\infty} \sum_{i=1}^{n} p_i(s)\, ds \leq 0,$$

which, in view of (12) and (13), implies a contradiction that complete our claim.

Next, to prove that $\limsup_{t \to \infty} z(t) = 0$, from (14) we can choose a subsequence $\{\xi_k\}_1^\infty$ from $\{t_k^*\}_m^\infty$ such that

$$\lim_{k \to \infty} z(\xi_k) = 0, \tag{17}$$

and similarly find another subsequence $\{\eta_k^+\}_1^\infty$ of $\{t_k^{*+}\}_m^\infty$ between ξ_k and ξ_{k+1}, $k = 1, 2, \ldots,$ such that $\lim_{k \to \infty} z(\eta_k^+) = \limsup_{t \to \infty} z(t)$. Assume that $b_k^*, _{\bar{k}}^*$ correspond to the moments ξ_k, η_k of impulsive effect, respectively. Then from (2) and (14), it follows from

$$0 < z(\eta_k^+) \leq (1 + b_{\bar{k}}^*) z(\eta_k) \leq (1 + b_{\bar{k}}^*) z(\eta_{\bar{k}-1}^+)$$
$$\leq (1 + b_{\bar{k}}^*)(1 + b_{\bar{k}-1}^*) \cdots (1 + b_k^*) z(\xi_k), \qquad k = 1, 2, \ldots,$$

and (17) that $\lim_{k \to \infty} z(\eta_k^+) = 0$. Therefore, $z(t) \to 0$ as $t \to \infty$. The proof of Theorem 5 is completed.

Theorem 1 in [7] is a special case of Theorem 5 here. Readers interested in it may combine it with Theorem 4 and obtain many results for themselves.

LEMMA 2. *Assume* (7) *and* (11) *hold. Let* $z(t)$ *be a nonoscillatory solution of Eq.* (1). *Let* $x(t)$ *be any oscillatory solution of Eq.* (2). *Then there exists* $K > 0$ *such that eventually*

$$|x(t)| \leq K |z(t)|.$$

Proof. Without loss of generality, suppose that $z(t) > 0$ for $t \geq T$. Hence, using the function w introduced in Lemma 1, we have to prove that w is bounded. Assume that it is not true.

From the definition of $w(t)$, it is easy to see that (11) holds if and only if

$$w(t_j)\big(w(t_j^+) - w(t_j)\big) \geq 0, \qquad j = 1, 2, \ldots .$$

Moreover, noting that w is an oscillatory function, there exists $t^* \geq T + \tau$ such that either

$$w(t^*) \leq 0 \quad \text{and} \quad w(t^*) > w(s) \qquad \text{for } T \leq s < t^*,$$

or

$$\dot{w}(t^*) \geq 0 \quad \text{and} \quad w(t^*) < w(s) \qquad \text{for } T \leq s < t^*,$$

in which t^* may be an element of $\{t_j^+\}$. Substituting $t^* = t$ into (6), Lemma 1 and (7) imply a contradiction.

COROLLARY 2. *Assume that* (4), (7), *and* (11)–(13) *are satisfied. Then every solution of Eq.* (2) *tends to zero as* $t \to \infty$.

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